ANALYSIS OF THE PROCESS OF BUCKLING OF RODS UPON IMPACT

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A lot of attention is being devoted at present to the study of the process of buckling of elastic systems upon the action of compressive loads of great strength. An explanation of the experimentally revealed phenomenon has been given by Lavrent'ev and Ishlinskii in [1], and it was shown that some higher mode of stability loss grows most intensely (exponentially). Since Lavrent'ev and Ishlinskii have used in their theoretical deliberations the simplest model of the bending of rods, the appearance of the investigations [2-9] was natural; a numerical and theoretical analysis is conducted in these references of the buckling of rods in which more exact theories are applied which take into account shear, rotational inertia, and finiteness of the propagation velocity of longitudinal disturbances. We note the experimental results of [10, 11], which have revealed the effect of the wave nature of the propagation of longitudinal disturbances on the distribution of deflections along the rod. All the investigations [2-11] have arrived at the conclusion that the number of the most intensely growing mode either coincides with or differs inappreciably from the number of the mode determined in [1]. The asymptotes of normal deflection of a rod at most times are obtained in [6-8] with longitudinal vibrations taken into account; in the limit of infinite propagation velocity of longitudinal disturbances the rate of increase of the deflections [6-8] does not agree with the conclusions of [1]. The number of the most intensely growing mode and the rate of increase of the deflections are refined below. The distribution of deflections along the rod is obtained in the case of a finite propagation velocity of longitudinal disturbances; in the limit when the velocity mentioned tends to infinity, the rate of increase of the deflections coincides with the results obtained in [1].

1. The system of equations which takes into account shear, rotational inertia, and the effect of longitudional vibrations on the transverse motions of rods is of the form (e.g., see [2, 12])

$$kFG(w_x-\psi)_x+EF[u_x(w_x+w_x^0)]_x+p(x,t)=\rho F w_{tt}; \tag{1.1}$$

$$EI\psi_{xx}+kFG(w_x-\psi)=\rho I\psi_{tt}; \tag{1.2}$$

$$EFu_{xx}=\rho F u_{tt}, \tag{1.3}$$

where u(x, t) and w(x, t) are the longitudinal and transverse displacements; $\psi(x, t)$, inclination angle of the tangent to the deflection curve; x and t, longitudinal coordinate and the time; E and G, moduli of elasticity and shear; F and I, area and moment of inertia of a transverse section of the rod; k, mode coefficient of the section; ρ, density of the material; p, transverse load; and w^0, initial deflection.

Let a compressive load N_0 significantly higher than the Euler load be applied to a flexibly fastened rod at rest at t = 0. The following boundary-value problem (l_0 is the length of the rod) is posed for the system (1.1)-(1.3):

$$w=0,\ \psi_x=0 \ \text{at} \quad x=0,\ l_0,\ t\geqslant 0; \tag{1.4}$$

$$w(x,0)=0,\ w_t(x,0)=0,\ \psi(x,0)=0,\ \psi_t(x,0)=0; \tag{1.5}$$

$$EFu_x(0,t)=-N_0,\ u(l_0,t)=0 \ \text{ or } \ u_x(l_0,t)=0; \tag{1.6}$$

$$u(x,0)=0,\ u_t(x,0)=0. \tag{1.7}$$

We will consider the time interval up to the first reflection of longitudinal waves from the end $x = l_0$. In this case the longitudinal forces are determined (see the problem (1.3), (1.6), and (1.7)) in the form

$$N(x,t)=EFu_x=-N_0\ x\leqslant ct,\ N(x,t)\equiv 0 \tag{1.8}$$

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with $ct < l_0$, where $c = (E/\rho)^{1/2}$ is the speed of sound. We will proceed to the problem (1.1), (1.2), (1.4), and (1.5). A theorem from [6] is used in connection with the solution of this problem; the theorem permits treating a linear system instead of the linear initial solution. For this linear system of equations in w and ψ after the introduction of the function $\Phi = \Phi(x, t)$

$$w = EI\Phi_{xx} - kFG\Phi - \rho I\Phi_{tt},\ \psi = -kFG\Phi_x \tag{1.9}$$

we obtain the solving equation

$$EI\Phi_{xxxx} - \rho I\frac{E+kG}{kG}\Phi_{xxtt} + N\left(\Phi_{xx} - \frac{EI}{kFG}\Phi_{xxxx} + \frac{\rho I}{kFG}\Phi_{xxtt}\right) + \rho F\Phi_{tt} + \frac{\rho^2 I}{kG}\Phi_{tttt} = -\frac{Nw_{xx}^0}{kFG} + \frac{p(x,t)}{kFG} \tag{1.10}$$

with the appropriate boundary and initial conditions. The conditions mentioned, which are obtained from (1.4) and (1.5) with (1.9) taken into account, are rather cumbersome, therefore they are not given here.

The dimensionless coordinates

$$x_1 = x/l_0,\ t_1 = ct/l_0$$

are introduced in Eq. (1.10) and the boundary and initial conditions. If we omit the index next to the new variables, we have finally the equation

$$\Phi_{xxxx} - m_1\Phi_{xxtt} + \pi^2\eta^2\Phi_{xx} - \pi^2 m_2\eta^2(r/l_0)^2\Phi_{xxxx} + \pi^2 m_2\eta^2(r/l_0)^2\Phi_{xxtt} + (l_0/r)^2\Phi_{tt} + m_2\Phi_{tttt} = f(x,t),$$

$$f(x,t) = \frac{l_0^4}{EI}\left(-\frac{Nw_{xx}^0}{kFG} + \frac{p(x,t)}{kFG}\right),\quad \eta^2 = -\frac{N}{P_e}, \tag{1.11}$$

$$\eta_0^2 = \frac{N_0}{P_e},\quad P_e = \frac{\pi^2 EI}{l_0^2},\quad m_1 = \frac{E+kG}{kG},\quad m_2 = \frac{E}{kG},$$

where $\eta = \eta(x, t)$ is a function; η_0, a parameter which characterizes the intensity of loading; P_e, Euler load; m_1 and m_2, parameters associated with taking the rotational inertia and shear into account; and r, radius of inertia of the transverse section. Those problems are being considered for which $\eta_0{}^2 = N_0/P_e \gg 1$. When $m_1 = m_2 = 0$, Eq. (1.11) changes into the classical equation of beam deflection.

2. First we will conduct an asymptotic analysis of Eq. (1.11) with the appropriate boundary and initial conditions for the simplest case, in which wave processes in the longitudinal direction are neglected, i.e.,

$$\eta^2 = \eta_0^2 = N_0/P_e,\quad 0 \leqslant x \leqslant l_0\quad (c \to \infty). \tag{2.1}$$

This problem permits separation of the variables ($X_n(x)$ are the stability loss modes)

$$\Phi = \sum_{n=1}^{\infty} T_n(t)\,X_n(x),\quad X_n(x) = \sin n\pi x,\quad n = 1, 2, \ldots \tag{2.2}$$

After simple rearrangements the following Cauchy problems (f_n are the coefficients of the Fourier series of the function f) are formulated for the amplitudes $T_n(t)$:

$$m_2(r/l_0)^2 T_n^{(4)} + [1 + \pi^2 m_1 n^2 (r/l_0)^2 - \pi^6 m_2 n^4\eta^2 (r/l_0)^4]\,T_n'' + (r/l_0)^2[\pi^4 n^4 - \pi^4 n^2\eta^2 - \pi^6 m_2 n^4\eta^2 (r/l_0)^2]\,T_n = f_n,$$
$$T_n(0) = 0,\quad T_n'(0) = 0,\quad T_n''(0) = 0,\quad T_n'''(0) = 0. \tag{2.3}$$

The equation of problem (2.3) has been written out in a form suitable for the asymptotic analysis. We note that when the simplest model of rod deflection [1] is used the coefficients $m_1 = m_2 = 0$. In each of the terms which contain the coefficients m_1 or m_2, the small parameter $\varepsilon = r/l_0 \ll 1$, which characterizes the relative length of the rod, is present. The small parameter ε enters to the second power into the coefficients of the fourth, second, and zeroth derivatives, but in addition to the fourth power for the second derivative. The term containing ε^4 is omitted in what follows. And so the equation of the problem (2.3) differs from the simplest equation in terms with small parameters both in the main part (the second and third terms) and of the leading derivatives (first term).

We will conduct an analysis and construct a solution of the Cauchy problem (2.3), which is perturbed singularly [13]. The solution of the problem (2.3) is separated into a smooth part $q_n(t)$ and a rapidly oscillating correction $s_n(t)$, so that

$$T_n(t) = q_n(t) + \varepsilon^{\alpha} s_n(t) = q_n^{(0)} + \varepsilon^2 q_n^{(2)} + \ldots + \varepsilon^{\alpha}\left[s_n^{(0)}(t) + \varepsilon^2 s_n^{(2)}(t) + \ldots\right]. \quad (2.4)$$

The value of the parameter $\alpha = 3$ is determined by the form of the boundary conditions of the problem (2.3); representations of the functions $q_n(t)$ to within an accuracy of ε^4 and of the functions $s_n(t)$ to within an accuracy of ε^2 are as follows:

$$q_n(t) = -(f_n/a_{0n})[\cosh(-a_{0n}/a_{2n})^{1/2}\, t - 1] \text{ for } a_{0n} < 0,$$
$$a_{2n} = 1 + \pi^2 m_1 n^2 (r/l_0)^2, \; q_n(t) \to \infty \text{ for } t \to \infty, \quad (2.5)$$
$$a_{0n} = (r/l_0)^2[\pi^4 n^4 - \pi^4 n^2 \eta^2 - \pi^6 m_2 n^4 \eta^2 (r/l_0)^2];$$
$$s_n(t)\left[f_n(-a_{0n})^{1/2} m_2^{2/3}/a_{2n}^{3/2}\right]\cos m_2^{-1/2}(l_0/r)\, t. \quad (2.6)$$

Equations (2.5) and (2.6) are derived for exponentially increasing solutions $T_n(t) \to \infty$ as $t \to \infty$; two other kinds of solution $T_n(t)$ are not given here, since one of them is a linear function of the time, and the other describes vibrations with a bounded amplitude. Following [1], we will determine the number n_* of the most intensely growing mode of motion, which with the specified loading intensity η is equal to

$$n_*^2 = \frac{\eta^2}{2}\frac{1}{1 - \pi^2 m_2 \eta^2 (r/l_0)^2} \quad (2.7)$$

to within an accuracy of ε^4.

And so when refined equations of the Timoshenko type are used, we see that for the motion growing most intensely in time (see (2.2), (2.4)-(2.7)) the mode number of buckling and the exponent for this motion are somewhat larger than the number and exponent for the motion predicted by the simplest theory [1], and the amplitude corresponding to this mode contains, in addition to the growing component [1], a rapidly oscillating component which is small in absolute value. The relationships obtained in the limit as $r/l_0 \to 0$ in (2.4) and (2.7) agree with the expressions given in [1].

3. Let us proceed to an analysis of the buckling process when the wave process of propagation of longitudinal vibrations with $c \neq \infty$ is taken into account, but only as far as the first reflection from the end $x = 1$. In solving the equation (1.11) with the constructed solution of propagation of the limiting conditions (1.8) taken into account we have for the dimensionless coordinates

$$\eta^2 = \eta_0^2 = N_0/P_e \quad \text{for} \quad x \leqslant t, \quad \eta^2 \equiv 0 \quad \text{for} \quad x > t. \quad (3.1)$$

If the transverse load $p(x, t)$ is such that $p(x, t) \equiv 0$ for $x > t$, then the maximum propagation velocity of disturbances for the system described by Eq. (1.11) with (3.1) taken into account coincides with the velocity of longitudinal waves in the rod, and consequently the conditions

$$\Phi - m_2(r/l_0)^2\Phi'' = 0, \; \Phi' = 0 \text{ at } x = t \quad (3.2)$$

are satisfied on the front. After the addition of the boundary conditions at $x = 0$ (e.g., conditions of flexible support) to the conditions (3.2) a problem on a variable interval [14] is obtained for Eq. (1.11). In order to analyze the process of buckling of a rod upon an intense longitudinal shock, the asymptote X_n of the natural modes of stability loss of a rod (Timoshenko's model) is investigated:

$$\{()'' + \lambda_n^2[1 - m_2(r/l_0)^2()'']\}\Phi'' = 0,$$
$$\Phi = \Phi'' = 0 \text{ at } x = 0, \; \Phi - m_2(r/l_0)^2\Phi'' = \Phi' = 0 \text{ at } x = 1. \quad (3.3)$$

If

$$1 - m_2(r/l_0)^2\lambda_n^2 > 0, \quad (3.4)$$

the asymptote of the eigenfunctions of the problem (3.3) as $n \to \infty$ coincides with the natural modes of stability loss of a flexibly supported rod $X(x) = \sin n\pi x$. The constraint (3.4) arises due to a certain inconsistency of Timoshenko's model for very short waves [5].

Let us carry further the analysis of the motions of a rod on a variable interval [5]. It has been established that those modes are developing most intensely which correspond to Eq. (2.7). Let us introduce the following transformation for Eq. (1.11) with (3.1) taken into account:

$$\tau = t - x. \tag{3.5}$$

The time coordinate is transformed such that it is converted into the true time of action of the compressive load in a fixed cross section of the rod. After the transformations (3.5) Eq. (1.11) contains terms with small factors, which we will omit in the following. We will restrict ourselves to the construction of the solution of a system with "one" degree of freedom which grows most rapidly [5]

$$\Phi(x,\ \tau) = T(\tau)X(x),$$
$$X(x) = \sin \pi x / l_* \text{ at } 0 \leqslant x \leqslant t,\ X(x) \equiv 0 \text{ at } x > t\ (x < 1).$$

Here l_* is the length of a halfwave corresponding to Eq. (2.7). The equation for the amplitude $T(\tau)$ has form (2.3) with null initial conditions; the form of the solution of this problem agrees with Eqs. (2.5) and (2.6) if one replaces t by τ in the latter and omits the index n. We will convert to the old coordinates (see (3.5)).

We obtain the solution in the form

$$\Phi(x,\ t) = \left\{ -\frac{f}{a_0}\left[\cosh\left(-\frac{a_0}{a_2}\right)^{1/2}(t-x) - 1\right] + \left(\frac{r}{l_0}\right)^3 \frac{f(-a_0)^{1/2} m_2^{2/3}}{a_2^{3/2}} \cos\frac{1}{m_2^{1/2}}\frac{l_0}{r}(t-x)\right\} \sin\frac{\pi x}{l_*} \text{ for } x \leqslant t, \tag{3.6}$$

$$\Phi(x,\ t) \equiv 0 \text{ for } x > t \quad (x < 1).$$

The solution obtained differs somewhat from the solution given in [5]: the wavelengths of stability loss and the rates of growth of the deflections agree to within an accuracy out to terms in small parameters, and the additional term in the braces of Eq. (3.6) has a small factor. The amplitude of the rapidly oscillating component of the solution (3.6) is small. When the finiteness of the propagation velocity of longitudinal disturbances is taken into account, neighboring halfwaves grow as if independently, but the amplitude of the deflections decreases exponentially from the impacted end of the rod.

Solution (3.6) given is in good agreement with natural experiments [10, 11] and with the results of numerical calculations given in [2]. These calculations have revealed the presence of a rapidly oscillating component of the solution whose amplitude is small. The computational results of [3, 4] given in graphical form are smoothed. After returning to dimensional coordinates in the representation of solution (3.6), a limiting transition such as $c \to \infty$ is possible. In this case a solution is obtained for a system with one degree of freedom; the wavelength and the growth rate of the deflections for this degree of freedom differ from those of [1] only in terms with small parameters. The solution (3.6) obtained does not agree with the results of [6-8].

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PROBLEM OF PROCESSING MATERIALS BY PRESSURE UNDER CREEPAGE CONDITIONS

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The pressure-processing of materials in the hot state is widely used in technology. The processing is usually not continuous, and deformation of the article is achieved due to "instantaneous" plastic deformation, while an increase in temperature, on the one hand, increases the plasticity of the material, and on the other reduces the force required for the deformation.

In recent years a considerable number of investigations have been made on the superplasticity of materials and the use of this phenomenon for the pressure-processing of materials. Superplastic behavior of a material is observed when it is in certain structural states, and in certain temperature ranges. But in all cases one of the decisive factors which facilitates superplastic deformation is slow loading. In such processes time plays an important part and deformation of creepage makes the main contribution to the total irreversible plastic deformations. Without dwelling on the physical reason and the similarities and differences in "instantaneous" plastic deformations and deformations of creepage, which develop with time, from the phenomenological point of view it can be stated that irreversible deformations determined by the laws of creepage are the initiating factors in superplasticity. In this sense, the hot processing of materials with slow loading should really be called "pressure-processing of materials under creepage conditions" [1]. Unlike the technological processes of processing materials in the superplastic state, processing under creepage conditions is less limited by technical difficulties and can be used in practice for all materials, including materials that are difficult to deform.

Publications on the use of creepage processes in the pressure-processing of materials have only appeared comparatively recently. In [2-4], neglecting the elastic-plastic components of the deformation and taking into account only deformation of creepage, solutions have been given of the problem of the sagging of a strip of a circular cylinder, longitudinal rolling, and a number of other problems encountered in technology. In [1, 5] some general considerations and experimental data on the possibility of using creepage in technological processes are presented, and the advantage of forming articles under slow rather than rapid loading conditions is pointed out. In [6] a description is given of a device which can be used in appropriate technological processes. But, on the whole, the number of papers on the experimental and theoretical principles of the use of creepage in the pressure-processing of materials is very small, which is undoubtedly the reason for the slow rate of development of this process.

Below we describe experiments which show the advantages of slow loading over fast loading. Using the generally accepted creep relations we give approximate methods of analyzing

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